Third isomorphism theorem

Statement :- If A & B are submodules of an R-module M such that $A \subset B$. Then

$$\frac{M}{B} \cong (M/A)\Big/(B/A)$$

i.e M/B is isomorphic to some quotient module of M/A

Theorem : Let A & B be R-submodules of modules M and N respectively. Then

$$\frac{M \times N}{A \times B} \cong \frac{M}{A} \times \frac{N}{B}$$

Proof : Consider the map

$$f : M \times N \to \frac{M}{A} \times \frac{N}{B} \text{ by}$$

$$f(m,n) = (m+A, n+B)$$

we will prove f is an onto R-homomorphism

i) f is ~~one-one~~ onto

Let $(m+A, n+B) \in \frac{M}{A} \times \frac{N}{B}$, then

$(m,n) \in M \times N$ is such that

$$f(m,n) = (m+A, n+B)$$

$\therefore$ f is onto.

ii) f is homomorphism.

let

$$f((m_1,n_1)+(m_2,n_2)) = f((m_1+m_2),(n_1+n_2))$$
$$= ((m_1+m_2)+A,(n_1+n_2)+B)$$
$$= ((m_1+A)+(m_2+A), (n_1+B)+(n_2+B))$$
$$= ((m_1+A),(n_1+B)) + ((m_2+A),(n_2+B))$$
$$= f(m_1,n_1) + f(m_2,n_2)$$

~~f(r(m₁,n~~

$$f(r(m_1,n_1)) = f(rm_1, rn_1)$$
$$= (rm_1+A, rn_1+B)$$
$$= r(m_1+A), r(n_1+B)$$
$$= r(m_1+A, n_1+B)$$
$$= r f(m_1,n_1)$$

$\therefore$ f is a homomorphism.

By F.T.H., we have

$$\frac{M\times N}{\ker f} \cong \frac{M}{A}\times\frac{N}{B}$$

where

$$\ker f = \{(m,n)\in M\times N \mid f(m,n)=(A,B)\}$$
$$= \{(m,n) \mid (m+A, n+B) = (A,B)\}$$
$$= \{(m,n) \mid m\in A \ \& \ n\in B\}$$
$$= A\times B$$

This completes the proof.

Example : Let R be a ring with unity. Then an R-module M is cyclic iff $M \cong R/I$ for some left ideal I of R.

Proof : Since R has unity, Suppose M is cyclic then $M = Rx$, for some $x \in M$.

Define, $f : R \to Rx$ by
$$f(r) = rx$$

It is easy to prove that f is an onto homomorphism.
So, by F.T.H.
$$\frac{R}{\ker f} \cong Rx$$
where
$$\ker f = \{ r \in R \mid f(r) = 0 \}$$
$$= \{ r \in R \mid rx = 0 \}$$

Then $\ker f = I$ is a left ideal of R

$$\therefore \quad R/I \cong Rx$$

Conversely, Suppose that $M \cong R/I$.

We note that the left R-module R/I is generated by $1 + I \in R/I$,

i.e $R(1+I) = R/I$ [$R(1+I)$ is a cyclic module generated by $(1+I)$]

Hence R/I is cyclic.

i.e. M is cyclic.

Example Let M be an R-module. Then show that $Hom_R(M,M)$ is a subring of $Hom(R,R)$.

Sol. Firstly we show that the set

$Hom(M,M)$ = the set of all group Homomorphism from abelian group M to M.

is a ring.

We define addition and multiplication in $Hom(M,M)$ as follows:

i) $(f+g)x = f(x)+g(x)$

ii) $fg(x) = f(g(x))$

$\forall \; f, g \in Hom(M,M)$ & $x \in M$.

(R,) $(Hom(M,M), +)$ is an abelian group

Closure property :- Let $f, g \in Hom(M,M)$ & $x, y \in M$. Then

$$(f+g)(x+y) = f(x+y) + g(x+y)$$

$$= [f(x)+f(y)] + [g(x)+g(y)]$$

[$\because$ f, g are homomorphism]

$$= [f(x)+g(x)] + (f(y)+g(y))$$

$$= (f+g)(x) + (f+g)(y)$$

This shows that $f+g \in Hom(M,M)$.

Commutatativity : Let $f, g \in Hom(M, M)$ & $x \in M$. Then

$$(f+g)(x) = f(x) + g(x)$$
$$= g(x) + f(x)$$
$$= (g+f)(x) \quad \forall x \in M$$

Therefore, $f+g = g+f \quad \forall f, g \in Hom(M, M)$

Associativity : Let $f, g, h \in Hom(M, M)$ & $x \in M$. Then

$$[(f+g)+h](x) = (f+g)(x) + h(x)$$
$$= [f(x) + g(x)] + h(x)$$
$$= f(x) + [g(x) + h(x)]$$

[because associative law for addition holds in M.]

$$= f(x) + (g+h)(x)$$
$$= [f + (g+h)](x)$$

Therefore $[(f+g)+h] = [f + (g+h)]$

$\forall f, g, h \in Hom(M, M)$

Existence of identity : The zero endomorphism of M defined by

$$\bar{0} : x \mapsto 0$$

is the additive identity of $Hom(M, M)$. For if $f \in Hom(M, M)$, then for any $x \in M$

$$(\bar{0}+f)(x) = \bar{0}(x) + f(x)$$
$$= 0 + f(x)$$
$$= f(x)$$

Thus $\bar{0}+f = f \quad \forall\ f \in Hom(M,M)$

Existence of inverse: If $f \in Hom(M,M)$ define $-f: M \to M$ by

$(-f)(x) = -f(x)\ \forall\ x \in M$.

Then $-f \in Hom(M,M)$ because, for any $x, y \in M$

$$(-f)(x+y) = -[f(x+y)]$$
$$= -[f(x)+f(y)]$$
$$= -f(x) - f(y)$$
$$= (-f)(x) + (-f)(y)$$

obviously, $(-f)+f = \bar{0} \quad \forall\ f \in Hom(M,M)$

R_2 $(Hom(M,M), \cdot)$ is a semi-group

Closure property: Let $f, g \in Hom(M,M)$ & let $x, y \in M$. Then

$$fg(x+y) = f(g(x+y))$$
$$= f[g(x)+g(y)]$$
$$= f(g(x)) + f(g(y))$$
$$= (fg)(x) + (fg)(y)$$

This show that

$f, g \in Hom(M,M) \Rightarrow fg \in Hom(M,M)$

Associativity Let $f, g, h \in Hom(M, M)$ & let $x \in M$. Then

$$\begin{aligned}[(fg)h](x) &= (fg)(h(x)) \\ &= f[g(h(x))] \\ &= f[gh(x)] \\ &= [f(gh)](x)\end{aligned}$$

Therefore $(fg)h = f(gh) \;\forall\; f, g, h \in Hom(M, M)$

R_3 Distributive law

For any $f, g, h \in Hom(M, M)$, we can easily see that

$(f+g)h = fh + gh$ [right distributive law]

$f(g+h) = fg + fh$ [left distributive law]

Also it is easily shown that the mapping

$1 : x \mapsto x$ [identity endomorphism of M]

of M into M is an endomorphism of M.

Hence $Hom(M, M)$ is a ring. Clearly

$Hom_R(M, M) \subset Hom(M, M)$

Next, we show that $Hom_R(M, M)$ is a subring of $Hom(M, M)$.

Let $f, g \in Hom_R(M, M)$, $x \in M$ and $r \in R$.

Then

$$(f-g)rx = f(rx) - g(rx)$$
$$= rf(x) - rg(x)$$
$$= r(f(x) - g(x))$$
$$= r(f-g)(x)$$

$$(fg)(rx) = f(g(rx))$$
$$= f(rg(x))$$
$$= rf(g(x))$$
$$= r(fg)(x)$$

Therefore $f, g \in Hom_R(M, M) \Rightarrow f-g, fg \in Hom_R(M, M)$

Hence, $Hom_R(M, M)$ is a subring of $Hom(M, M)$

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$RM = \{ \sum r_i m_i \mid r_i \in R \ \& \ m_i \in M \}$

Then

1) $RM = \{\bar{0}\} \not\Rightarrow M = \{\bar{0}\}$, But
ii) if $1 \in R$, then $RM = \{0\} \Rightarrow M = \{0\}$

iii) The Submodule $M = \{\bar{0}\}$ are called improper Submodulet and any submodule N s.t.

$\{0\} \neq N \neq M$, is called proper Submodule.

Simple Module / Irreducible Module

Definition : An R-module M is Said to be Simple or irreducible if $RM \neq \{0\}$ and M and $\{0\}$ are only submodules of M i.e. M has no proper submodule.

Example : (1) Note that

division ring + commutativity = field.

1) Every division ring and field R are Simple modules
2) If R is a division ring then R is a R-module

$$RR = R \neq \{0\} \quad (\text{Here } M = R)$$

all the Submodule of a ring R, regarded as R-module, are precisely the left ideals of R. Since R is a division ring so every non-zero element of R has inverse.

If I is any ideal & $I \neq \{\bar{0}\}$ then $\exists$ $0 \neq a \in I$ then $a^{-1} \in R$ is such that

$$a^{-1}a = 1 \in I \quad \{\text{By def. of ideal}\}$$

$\Rightarrow$ $r \cdot 1 = r \in I \quad \forall\ r \in R$

i.e $R \subset I$

$\therefore$ $I = R$

Hence, as a ring R, has only two left ideals R & $\{\bar{0}\}$.

$\Rightarrow$ As module R has only two R-submodules R & $\{0\}$

$\Rightarrow$ R is simple.

Characterization of Simple R-module

Theorem : Let R be a ring with unity and M be an R-module. Then the following statement are equivalent

i) M is simple

ii) $M \neq \{0\}$ and M is generated by any $0 \neq x \in M$.

iii) $M \cong R/I$, where I is a maximal left ideal of R.

Proof :- (i) $\Rightarrow$ (ii)

Since R has unity & M is simple by (i)
Then $RM = M \neq (0)$ and (0), M are the only R-submodules of M.
Thus, we required to show that M is generated by any $0 \neq x \in M$.
So, let $0 \neq x \in M$. Then
$(x) = Rx$ is a non-zero R-submodule generated by x.

Hence $M = (x)$

(ii) $\Rightarrow$ (i)

Assume that $M \neq (0)$ and $M = (x)$ for any $0 \neq x \in M$.
Then we have to show that M is simple.

Let $\{0\} \neq N$ be any R-submodule of M.
Then there exists $0 \neq x \in N$, and we have $Rx \subset N$ and by (ii) $M = Rx$
$\Rightarrow M \subset N$

But, clearly $N \subset M$, whereby $N = M$
This proves that M is simple.
Hence (ii) $\Rightarrow$ (i) is established.

(i) $\Rightarrow$ (iii)

Since R has unity & M is simple by (i) we have $M \neq \{0\}$
So, there exists $0 \neq x \in M$. Then

$$Rx = \{rx \mid r \in R\} \text{ is an } R\text{-Submodule generated by } x.$$

But, M is simple so the only submodules of M are $\{0\}$ & M.

If $Rx = \{0\} \Rightarrow x = 0$, contradicts our choice of x.

Therefore, $Rx = M$

Now, define a map

$$f : R \to Rx \text{ by}$$
$$f(r) = rx \;\; \forall\, r \in R$$

Then (easy to see) f is a R-homomorphism which is onto. So, by Fundamental theorem of Homomorphism

$$R/\ker f \cong Rx$$

where $\ker f = \{r \in R \mid f(r) = 0\}$
$= \{r \in R \mid rx = 0\}$

which is a left ideal of R, say I, i.e

$$I = \ker f$$

Then $R/I \cong Rx = M$ $(\because M = Rx)$

Next, ~~Simple~~ M is simple, so R/I is also simple R-module

Therefore, only left R-submodules of R/I are $\{I\}$ and R/I

Since, every submodule of R/I is of the form U/I, where U is some left ideal of R containing I.

But, since R/I is simple, so R/I has no proper submodules

$\Rightarrow$ there is no left ideal U between I and R i.e $I \subsetneq U \subsetneq R$

$\Rightarrow$ I is maximal left ideal

Hence $M \cong R/I$, where I is a maximal left ideal of R

This proves (i) $\Rightarrow$ (iii)

(iii) $\Rightarrow$ (i)

By (iii) $M \cong R/I$, where I is maximal left ideal of R

Since, I is maximal left ideal of R.
So, there is no left ideal U between I & R
i.e. no U such that

$$I \subsetneq U \subsetneq R$$

$\Rightarrow$ R/I has no proper R-Submodules

$\Rightarrow$ R/I is simple

$\Rightarrow$ M is simple $\quad (\because M \cong R/I)$

This proves (iii) $\Rightarrow$ (i)

Exact Sequence

Definition : A Sequence (finite or infinite) of R-modules and R-homomorphisms

$$\cdots \xrightarrow{f_{n-1}} M_{n-1} \xrightarrow{f_n} M_n \xrightarrow{f_{n+1}} M_{n+1} \xrightarrow{f_{n+2}} \cdots$$

is said to be exact if

$$Im(f_n) = ker(f_{n+1}) \quad \forall n$$

Def: A diagram commutes if $h = g \circ f$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$