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Poisson's Distribution as a Limiting form of Binomial Distribution :

Poisson distribution is a particular limiting form of the binomial distribution when p (or q) is a very small and n is large so that the average number of successes np is a finite constant m (let)

We know that, in the binomial distribution, the probability of r success is given by

$$b(r, n, p) = {}^{n}C_r \, p^r q^{n-r}$$

$$= {}^{n}C_r \, p^r (1-p)^{n-r} \qquad [\because p+q=1]$$

$$= \frac{n(n-1)(n-2)\cdots(n-r+1)}{\lfloor r} \, p^r \left(1-\frac{np}{n}\right)^{n-r}$$

$$= \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{r-1}{n}\right)}{\lfloor r} \, \frac{(np)^r}{\left(1-\frac{np}{n}\right)^r} \left(1-\frac{np}{n}\right)^n$$

$\therefore P(r) =$ The probability of r successes in Poisson's distribution.

$$= \lim_{p\to 0,\, n\to\infty,\, np=m} b(r, n, p)$$

$$= \lim_{n\to\infty} \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{r-1}{n}\right)}{\lfloor r} \, \frac{m^r}{\left(1-\frac{m}{n}\right)^r} \left(1-\frac{m}{n}\right)^n$$

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$$= \frac{m^r e^{-m}}{\lfloor r} \left[\because \lim_{n\to\infty} \left(1-\frac{m}{n}\right)^n = \lim_{n\to\infty} \left\{ \left(1-\frac{m}{n}\right)^{-\frac{n}{m}} \right\}^{-m} = e^{-m} \right]$$

and $\lim_{n\to\infty} \left(1-\frac{m}{n}\right)^r = (1-0)^r = 1$

is called Poisson distribution.

Therefore the chances of $0, 1, 2, \ldots r$ successes are respectively

$$e^{-m}, \frac{me^{-m}}{\lfloor 1}, \frac{m^2 e^{-m}}{\lfloor 2} \ldots \frac{m^r e^{-m}}{\lfloor r}$$

$\therefore$ The limiting form of binomial distribution $(p+q)^n$ when $p \to 0, n \to \infty$

So that $np = m$ is called Poisson distribution.

Definition: The Probability distribution of a random variable x is called Poisson's distribution if x can assume non-negative values only and its distribution is given by

$$P(X=r) = \begin{cases} \frac{e^{-m} m^r}{\lfloor r} & r = 0, 1, 2 \ldots \\ 0 & r \neq 0, 1, 2 \ldots \end{cases}$$

m is known as parameter of poisson's distribution

Examples of Poisson's distribution

(i) The number of deths in a city in one year by a rat disease or by heart attack or cancer

(ii) The number of telephone calls per minuts in a switch board.

(iii) The number of suicides in a city in one year

Moment's about the Origin

(i) First moment about the origin

$$\mu_1' = \sum_{r=0}^{\infty} e^{-m} \frac{m^r}{\lfloor r} \cdot r = \sum_{r=0}^{\infty} \frac{e^{-m} m^r}{\lfloor r-1}$$

$$= e^{-m}\left(m + \frac{m^2}{\lfloor 1} + \frac{m^3}{\lfloor 2} + \cdots\right)$$

or $\mu_1' = m e^{-m}\left(1 + \frac{m}{\lfloor 1} + \frac{m^2}{\lfloor 2} + \cdots\right)$

$$= m e^{-m} e^{m} = m$$

i.e mean $= \mu_1' = m$

(ii) Second moment about the Origin

$$\mu_2' = \sum_{r=0}^{\infty} e^{-m} \frac{m^r}{\lfloor r} r^2.$$

$$= \sum_{r=0}^{\infty} e^{-m} \frac{m^r}{\lfloor r} \{r(r-1) + r\}$$

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$$= \sum_{r=0}^{\infty} \frac{e^{-m} m^r}{\lfloor r-2} + \sum_{r=0}^{\infty} \frac{e^{-m} m^r}{\lfloor r-1}$$

$$= \sum_{r=0}^{\infty} \frac{m^2 e^{-m} m^{r-2}}{\lfloor r-2} + \sum_{r=0}^{\infty} \frac{m e^{-m} m^{r-1}}{\lfloor r-1}$$

$$= m^2 e^{-m} \sum_{r=0}^{\infty} \frac{m^{r-2}}{\lfloor r-2} + m e^{-m} \sum_{r=0}^{\infty} \frac{m^{r-1}}{\lfloor r-1}$$

$$= m^2 e^{-m} e^{m} + m e^{-m} e^{m}$$

$$= m^2 + m$$

(iii) Third moment about the origin

$$\mu_3' = \sum_{r=0}^{\infty} e^{-m} \frac{m^r}{\lfloor r} \cdot r^3$$

$$= \sum_{r=0}^{\infty} \left[e^{-m} \frac{m^r}{\lfloor r} \left\{ r(r-1)(r-2) + 3r(r-1) + r \right\} \right]$$

$$= m^3 + 3m^2 + m.$$

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Example: Fit Poisson's Destribution to the following and calculate theoretical frequencies $e^{-0.5} = 0.61$

Deth :	0	1	2	3	4
Frequency :	122	60	15	2	1

Solution: Here total frequency

$(N) = \Sigma f = 122 + 60 + 15 + 2 + 1 = 200$

$$\text{Mean } (m) = \frac{\Sigma f(x)}{N}$$

$$= \frac{122 \times 0 + 60 \times 1 + 15 \times 2 + 2 \times 3 + 1 \times 4}{200}$$

$$= \frac{60 + 30 + 6 + 4}{200} = \frac{1}{2} = 0.5$$

Now $e^{-m} = e^{-0.5}$

$$= 1 - (0.5) + \frac{1}{2}(0.5)^2 - \frac{1}{6}(0.5)^3 + \cdots$$

$$= 1 - 0.5 + 0.125 - 0.0208 + \cdots$$

$$= 0.61 \text{ (approx)}$$

$\therefore$ Therefore theoretical frequency of r deths

$$= N \cdot e^{-m} \frac{m^r}{\lfloor r} = 200 \times (0.61) \times \frac{(0.5)^r}{\lfloor r}$$

computation of theoretical frequencies

x	$P(X=x)$	$N \cdot P(X=x)$ Expected frequency
0	$e^{-m} = 0.61$	$200 \times 0.61 = 122$
1	$me^{-m} = 0.305$	$200 \times 0.305 = 61$
2	$\frac{m^2}{\lfloor 2} e^{-m} = 0.0762$	$200 \times 0.0762 = 15$
3	$\frac{m^3}{\lfloor 3} e^{-m} = 0.0127$	$200 \times 0.127 = 2$
4	$\frac{m^4}{\lfloor 4} e^{-m} = 0.0016$	$200 \times 0.0016 = 0$
Total	$1.0055 = 1$	200

Thus it gives frequencies 122, 61, 15, 2, and 0 respectively for $x = 0, 1, 2, 3,$ and 4

Example: Find the probability that at most 5 defective fuses will be found in a box of 200 fuses, if experience shows that 2 percent of such ~~of~~ fuses are defective.

Solution: Here $p = 2\% = 2/100 = 1/50$

$n = 200$

$\therefore np = 200/50 = 4.$

Maximum number of defective fuses $= 5$

$\therefore r \leq 5$

Also $e^{-4} = 1 - 4 + \frac{1}{2} 4^2 - \frac{1}{\lfloor 3} 4^3 + \frac{1}{\lfloor 4} \cdot 4^4$

$= 0.0183$

$\therefore$ Required Probability $(r \leq 5)$

$$= \sum_{r=0}^{5} \frac{4^r e^{-4}}{\lfloor r}$$

$$= e^{-4} \left[1 + \frac{4}{\lfloor 1} + \frac{4^2}{\lfloor 2} + \frac{4^3}{\lfloor 3} + \frac{4^4}{\lfloor 4} + \frac{4^5}{\lfloor 5} \right]$$

$= 0.0183 \times 42.8667 = 0.7845$