Example. 2. Find the complete integral of $(p^2+q^2)y = qz$.

Solve by similar to exp. 1.

Answer. $z^2 = a^2y^2 + (ax+b)^2$

Example. 3. Find a C.I of $p^2+q^2-2px-2qy+2xy = 0$.

Solution. Here $f = p^2+q^2-2px-2qy+2xy = 0$ — (1)

Now find $\frac{\partial f}{\partial x} = -2p+2y$, $\frac{\partial f}{\partial y} = -2q+2x$,

$\frac{\partial f}{\partial z} = 0$, $\frac{\partial f}{\partial p} = 2p-2x$, $\frac{\partial f}{\partial q} = 2q-2y$.

Putting these values in Charpit's Auxiliary Equations

$$\frac{dp}{\partial f/\partial x + p\,\partial f/\partial z} = \frac{dq}{\partial f/\partial y + q\,\partial f/\partial z} = \frac{dx}{-\partial f/\partial p} = \frac{dy}{-\partial f/\partial q}$$

$$= \frac{dz}{-p\,\partial f/\partial p - q\,\partial f/\partial q}$$

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$$\Rightarrow \frac{dp}{-2p+2y} = \frac{dq}{-2q+2x} = \frac{dx}{-(2p-2x)} = \frac{dy}{-(2q-2y)}$$

$$= \frac{dz}{-2p^2+2px-2q^2+2qy}$$

$$\Rightarrow \cancel{\frac{dp}{-2p+2y}} = \cancel{\frac{dq}{-2q+2x}} = \cancel{\frac{dx}{-(2p-2x)}} = \frac{dy}{}$$

$$\Rightarrow \frac{dp}{y-p} = \frac{dq}{x-q} = \frac{dx}{-(p-x)} = \frac{dy}{-(q-y)}$$

$$= \frac{dz}{px+qy-p^2-q^2}$$

How using multiplyer method

Each ratio $= \frac{dp+dq}{x+y-p-q} = \frac{dx+dy}{-(p+q-x-y)}$

taking $\frac{dp+dq}{x+y-p-q} = \frac{dx+dy}{(x+y-p-q)}$

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we get $dp + dq = dx + dy$

Integrating, $p + q = x + y + a$ — (2)

Again equation (1) can be written as

$(p-x)^2 + (q-y)^2 = (x-y)^2$ — (3)

Now solving (2) and (3) we have

put $p - x = P$ and $q - y = Q$ in (2) & (3)

So that $P + Q = a$ and $P^2 + Q^2 = (x-y)^2$

we know that

$$(P-Q)^2 = (P^2+Q^2) - 2PQ$$

$$= P^2 + Q^2 - \{(P+Q)^2 - (P^2+Q^2)\}$$

$$= 2(P^2+Q^2) - (P+Q)^2$$

$$P - Q = \sqrt{2(x-y)^2 - a^2}$$

& $P + Q = a$

$$\therefore P = \frac{1}{2}\left[a + \sqrt{2(x-y)^2 - a^2}\right] = p - x$$

$$\& \quad Q = \frac{1}{2}\left[a - \sqrt{2(x-y^2) - a^2}\right] = q - y$$

$$\therefore p = x + \frac{1}{2}\left[a + \sqrt{2(x-y)^2 - a^2}\right]$$

and $q = y + \frac{1}{2}\left[a - \sqrt{2(x-y)^2 - a^2}\right]$

Putting the values of p and q in $dz = p\,dx + q\,dy$

$$\Rightarrow dz = \left[x + \frac{1}{2}\left(a + \sqrt{2(x-y)^2 - a^2}\right)\right] dx + \left[y + \frac{1}{2}\left(a - \sqrt{(x-y)^2 - a^2}\right)\right] dy$$

$$dz = \left(x + \frac{a}{2}\right)dx + \left(y + \frac{a}{2}\right)dy + \frac{1}{2}\sqrt{2(x-y)^2 - a^2}\,(dx - dy)$$

$$2\,dz = (2x + a)\,dx + (2y + a)\,dy + \sqrt{u^2 - a^2}\,\frac{du}{\sqrt{2}}$$

$$\left[\text{where } \sqrt{2}(x-y) = u.\right]$$

Integrating both side we get

$$2z = x^2 + ax + y^2 + ay + \frac{1}{\sqrt{2}}\left(\frac{u}{2}\sqrt{u^2 - a^2} - \frac{a^2}{2}\log\left\{u + \sqrt{u^2 - a^2}\right\}\right) + b$$

which is a complete integral of (2) where $u = \sqrt{2}(x-y)$.

Example. 4. Solve by Charpit's method:

$$pxy + pq + qy = yz.$$

Sol. Solve. Similar to Example. 3.

Example. 5. Use Charpit's method to find complete integral of

$$pq = px + qy.$$

Sol. Let $f = pq - px - qy = 0$ (1)

$$\frac{\partial f}{\partial x} = -p, \quad \frac{\partial f}{\partial y} = -q, \quad \frac{\partial f}{\partial z} = 0, \quad \frac{\partial f}{\partial p} = q - x,$$

$$\frac{\partial f}{\partial q} = p - y$$

Putting these value in Charpit's auxiliary equation -

$$\frac{dp}{-p + p\cdot 0} = \frac{dq}{-q + q\cdot 0} = \frac{dx}{x - q} = \frac{dy}{y - p} =$$

$$\frac{dz}{-p(q - x) - q(p - y)}$$

taking first two ratio

$$\frac{dp}{+p} = \frac{dq}{+q}$$

Integrating

$$\log p = \log q + \log a$$

$$\log p = \log (aq)$$

$$p = aq \quad — (2)$$

from equation (i) we have

$$aq^2 - aqx - qy = 0$$

$$\Rightarrow q = \frac{ax+y}{a} \quad —(3)$$

From (2) and (3) we get

$$p = ax+y$$

Putting the values of p and q in

$$dz = p\,dx + q\,dy$$

$$dz = (ax+y)\,dx + \left(\frac{ax+y}{a}\right)dy$$

$$dz = \frac{1}{a}(ax+y)(a\,dx+dy)$$

$$dz = \frac{1}{a}u\,du \Rightarrow z = \frac{u^2}{2a} + b$$

Put $ax+y = u$

$a\,dx + dy = du$

$\therefore z = \frac{1}{2a}(ax+y)^2 + b$ which is required complete integral.

Example-6. Solve $p = (z+qy)^2$

Sol. Let $f \equiv (z+qy)^2 - p = 0$ — (1)

Now find $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 2q(z+qy)$,

$\frac{\partial f}{\partial z} = 2(z+qy)$, $\frac{\partial f}{\partial p} = -1$,

$\frac{\partial f}{\partial q} = 2y(z+qy)$

Putting these values in Charpit's Auxiliary equation

$$\therefore \frac{dx}{1} = \frac{dy}{-2y(z+qy)} = \frac{dz}{p-2qy(z+qy)} = \frac{dp}{0+2p(z+qy)}$$

$$= \frac{dq}{4q(z+qy)}$$

Taking second and fourth ratio we get

$$\frac{dy}{-2y(z+qy)} = \frac{dp}{2p(z+qy)}$$

$$\Rightarrow \frac{dp}{p} = \frac{dy}{-y}$$

Integrating we get

$$\log p = -\log y + \log a \Rightarrow \boxed{p = a/y}$$

Putting the value of p in equation (1) we get

$$(z + qy)^2 = \frac{a}{y} \Rightarrow q = \frac{\sqrt{a}}{y^{3/2}} - \frac{z}{y}$$

Now substituting the values of p and q into

$$dz = p\,dx + q\,dy$$

$$dz = \frac{a}{y}\,dx + \left(\frac{\sqrt{a}}{y^{3/2}} - \frac{z}{y}\right) dy$$

$$y\,dz = a\,dx + \sqrt{\frac{a}{y}}\,dy - z\,dy$$

$$y\,dz + z\,dy = a\,dx + \sqrt{a} \cdot y^{-1/2}\,dy$$

Integrating both side, we get

$$yz = ax + \sqrt{a}y + b.$$

which is required complete integral.