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Example-6. Solve $p = (z+qy)^2$

Sol. Let $f \equiv (z+qy)^2 - p = 0$ — (1)

Now find $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 2q(z+qy)$,

$$\frac{\partial f}{\partial z} = 2(z+qy), \quad \frac{\partial f}{\partial p} = -1,$$

$$\frac{\partial f}{\partial q} = 2y(z+qy)$$

Putting these values in Charpit's Auxiliary equation

$$\therefore \frac{dx}{1} = \frac{dy}{-2y(z+qy)} = \frac{dz}{p-2qy(z+qy)} = \frac{dp}{0+2p(z+qy)}$$

$$= \frac{dq}{4q(z+qy)}$$

Taking second and fourth ratio we get

$$\frac{dy}{-2y(z+qy)} = \frac{dp}{2p(z+qy)}$$

$$\Rightarrow \frac{dp}{p} = \frac{dy}{-y}.$$

Integrating we get

$$\log p = -\log y + \log a \Rightarrow \boxed{p = a/y}.$$

Putting the value of p in equation (1) we get

$$(z + qy)^2 = \frac{a}{y} \Rightarrow q = \frac{\sqrt{a}}{y^{3/2}} - \frac{z}{y}$$

Now substituting the values of p and q into

$$dz = p\,dx + q\,dy$$

$$dz = \frac{a}{y}dx + \left(\frac{\sqrt{a}}{y^{3/2}} - \frac{z}{y}\right)dy$$

$$y\,dz = a\,dx + \sqrt{\frac{a}{y}}\,dy - z\,dy$$

$$y\,dz + z\,dy = a\,dx + \sqrt{a}\cdot y^{-1/2}\,dy$$

Integrating both side, we get

$$yz = ax + \sqrt{a}y + b.$$

which is required complete integral.

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Example 7. Find the C.I. of the partial differential equation
$$p^2x + q^2y = z.$$

Sol. Let $f \equiv p^2x + q^2y - z = 0$ — (1)

Now find $\frac{\partial f}{\partial x} = p^2$; $\frac{\partial f}{\partial y} = q^2$,

$$\frac{\partial f}{\partial z} = -1, \quad \frac{\partial f}{\partial p} = 2px, \quad \frac{\partial f}{\partial q} = 2qy.$$

putting these values in Charpit's auxiliary equ^n.

$$\frac{dx}{-2px} = \frac{dy}{-2qy} = \frac{dz}{-2p^2x - 2q^2y} = \frac{dp}{p^2 - p} = \frac{dq}{q^2 - q}$$

Now using multiplier,

$$\text{each ratio} = \frac{p^2dx + 2px\,dp}{-2p^2x + 2px(p^2 - p)} = \frac{q^2dy + 2qy\,dq}{-2q^2y + 2qy(q^2 - q)}$$

$$= \frac{p^2dx + 2px\,dp}{-\cancel{2}p^2x} = \frac{q^2dy + 2qy\,dq}{-\cancel{2}q^2y}$$

taking above ratio we have.

$$\frac{p^2 dx + 2px\,dp}{p^2 x} = \frac{q^2 dy + 2qy\,dq}{q^2 y}$$

Integrating,

$$\log(p^2 x) = \log(q^2 y) + \log a.$$

$$p^2 x = a q^2 y \Rightarrow p^2 = a q^2 y / x.$$

putting the values of p in (1) we have

$$a q^2 y + q^2 y = z \Rightarrow q^2 y = \frac{z}{1+a}$$

$$\Rightarrow q = \sqrt{\frac{z}{(1+a)y}}$$

and $p^2 = ay \cdot \frac{z}{(1+a)\cancel{y}x} \Rightarrow p = \sqrt{\frac{az}{(1+a)x}}.$

Now putting the values of p and q in equation

$$dz = p\,dx + q\,dy$$

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$$dz = \sqrt{\frac{az}{(1+a)x}}\,dx + \sqrt{\frac{z}{(1+a)y}}\,dy$$

$$\sqrt{(1+a)} \cdot \frac{dz}{\sqrt{z}} = \sqrt{a} \cdot \frac{dx}{\sqrt{x}} + \frac{dy}{\sqrt{y}}$$

Integrating we get

$$2\sqrt{(1+a)z} = 2\sqrt{ax} + 2\sqrt{y} + 2b$$

$$\Rightarrow \sqrt{(1+a)z} = \sqrt{ax} + \sqrt{y} + b$$

which is required complete integral.

Example-8. Solve $p^2 + q^2 - 2pq\tanh(2y) = \operatorname{sech}^2(2y)$

Sol. let $f = p^2 + q^2 - 2pq\tanh(2y) - \operatorname{sech}^2(2y) = 0$ —(1)

Now find $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = -2pq\operatorname{sech}^2(2y)\cdot 2 - 4\operatorname{sech}^2(2y)\tanh(2y)$

$\frac{\partial f}{\partial z} = 0$, $\frac{\partial f}{\partial p} = 2p - 2q\tanh(2y)$,

$\frac{\partial f}{\partial q} = 2q - 2p\tanh(2y)$

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putting these values in Charpit's Auxiliary Equation

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}} = \frac{dx}{-\partial f/\partial p} = \frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}$$

$$\Rightarrow \frac{dp}{0} = \frac{dq}{-4pq\,\mathrm{sech}^2(2y)+4\,\mathrm{sech}^2(2y)\tanh(2y)}$$

$$= \frac{dx}{-2p+2q\tanh(2y)} = \frac{dy}{-2q+2p\tanh(2y)} =$$

$$\frac{dz}{-2p^2+2pq\tanh(2y)-2q^2+2pq\tanh(2y)}$$

taking $dp=0 \Rightarrow p=a$.

from equation (1) we have

$$a^2+p^2-2aq\tanh(2y)-\mathrm{sech}^2(2y)=0.$$

$$\Rightarrow q = \frac{2a\tanh(2y)\pm\sqrt{4a^2\tanh^2(2y)-4(a^2-\mathrm{sec}}}{2\times 1}$$

$$= a\tanh(2y)\pm\sqrt{a^2(\tanh^2 2y+\mathrm{sech}^2 2y}$$

$= a\tanh(2y) \pm \sqrt{-a^2(\tanh^2 2y - 1) + \operatorname{sech}^2 2y}$

$= a\tanh(2y) \pm \sqrt{-a^2 \operatorname{sech}^2 2y + \operatorname{sech}^2 2y}$

$q = a\tanh(2y) \pm \sqrt{(1-a^2)}\operatorname{sech}(2y)$

putting the values of p and q in $dz = p\,dx + q\,dy$

$dz = a\,dx + [a\tanh(2y) \pm \sqrt{1-a^2}\operatorname{sech}(2y)]\,dy$

Integrate,

$z = ax + \frac{a}{2}\log\cosh(2y) \pm \sqrt{1-a^2}\int\frac{2}{e^{2y}+e^{-2y}}\,dy + b$

$= ax + \frac{a}{2}\log\cosh(2y) \pm \sqrt{1-a^2}\int\frac{2e^{2y}}{1+e^{4y}}\,dy + b$

$= ax + \frac{a}{2}\log\cosh(2y) \pm \sqrt{1-a^2}\cdot\tan^{-1}(e^{2y}) + b$

which is complete integral of given equation.

Example. 9. Solve by Charpit's Method:

$$2z + p^2 + qy + 2y^2 = 0 \quad -(1)$$

Sol. Let $f = 2z + p^2 + qy + 2y^2 = 0$

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = q + 4y, \quad \frac{\partial f}{\partial z} = 2z$$

$$\frac{\partial f}{\partial p} = 2p, \quad \frac{\partial f}{\partial q} = y$$

putting these values in Charpit's Auxiliary equation.

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}}$$

$$= \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\Rightarrow \frac{dp}{2p} = \frac{dq}{q + 4y + 2q} = \frac{dz}{-2p^2 - qy} = \frac{dx}{-2p} = \frac{dy}{-y}$$

taking Ratio's

$$\frac{dp}{2p} = \frac{dx}{-2p} \Rightarrow dp = -dx.$$

Integrating both sides we get

$$p = -x + a \Rightarrow p = a - x \quad \text{--- (2)}$$

from (1) and (2) we have.

$$q = -\frac{1}{y}\{(a-x)^2 + 2z + 2y^2\}.$$

putting the value of p and q in equation.

$$dz = p\,dx + q\,dy$$

$$\Rightarrow dz = (a-x)\,dx - \frac{1}{y}\{(a-x)^2 + 2z + 2y^2\}\,dy$$

$$\Rightarrow 2y^2 dz = 2(a-x)y\,dx - 2y(a-x)^2 dy - 4yz\,dy - 4y^3 dy$$

$$\Rightarrow 2y^2 dz + 4yz\,dy = 2(a-x)y\,dx - 2y(a-x)^2 dy - 4y^3 dy$$

$$\Rightarrow d(2y^2 z) = -d\{y^2(a-x)^2\} - d(y^4)$$

Integrated,

$$2y^2 z = -y^2(a-x)^2 - y^4 + b$$

$$\Rightarrow y^2\{2z + (a-x)^2 + y^2\} = b$$

which is required C.I.

Some Practice Problems.

Solve the following problems by using Charpit's method.

(1) $p^2 - y^2 q = y^2 - x^2$

(2) $z^2 = pqxy$.

(3) $q = 3p^2$

(4) $pxy + pq + qy = yz$.

(5) $f(p,q) = 0$.