Gradient, Divergence and Curl

The operator ∇ (del operator)

The operator (vector) ∇ is defined by the equation

$$\nabla \equiv i\frac{\partial}{\partial x} + J\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

Hence $\underset{\text{grad } \phi}{\underset{||}{\nabla \phi}} = i\frac{\partial \phi}{\partial x} + J\frac{\partial \phi}{\partial z} + k\frac{\partial \phi}{\partial z}$ where ϕ is any scalar

To express the directional derivative in terms ∇

The directional derivative of ϕ along the direction of unit vector a is given by

$$\frac{d\phi}{ds} = \hat{a}\cdot \text{grad } \phi = \hat{a} \nabla \phi$$

Gradient of the product of two functions

If f and g are two scalar point functions then

$$\text{grad}(fg) = f \text{ grad } g + g \text{ grad } f$$

$$\nabla (fg) = f\nabla g + g \nabla f$$

Gradient of the Quotient of two functions

If f and g are two scalar point functions then

$$\text{grad}\left(\frac{f}{g}\right) = \frac{g \text{ grad } f - f \text{ grade } g}{g^2}, \quad g \neq 0$$

$$\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}, \quad g \neq 0$$

gradient of a constant

If ϕ is constant Then $\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}$ are all zero Therefore

$$\text{grad } \phi = i \frac{\partial \phi}{\partial x} + J \frac{\partial \phi}{\partial y} + K \frac{\partial \phi}{\partial z}$$

$$\text{grad } \phi = i(0) + J(0) + K(0)$$

$$\text{grad } \phi = \hat{0}$$

Result: $\bar{r} = \hat{r} r$

Question 1: If $\hat{r} = xi + yj + zk$ Then show that

$\text{grad } r = \hat{r}$ (or $\nabla r = \hat{r}$) where $\text{grad } r = |r|$

Solution: Since $r^2 = x^2 + y^2 + z^2$ we have

$$\frac{\partial r}{\partial x} = 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \quad \text{— (i)}$$

Similarly

$$\frac{\partial r}{\partial y} = \frac{y}{r} \quad \text{— (ii)}$$

$$\frac{\partial r}{\partial z} = \frac{z}{r} \quad \text{— (iii)}$$

Hence $\text{grad } r = \nabla r = i\frac{\partial r}{\partial x} + j\frac{\partial r}{\partial y} + k\frac{\partial r}{\partial z}$

$= i\frac{x}{r} + j\frac{y}{r} + k\frac{z}{r}$ from (1) (2) & (3)

$= \frac{xi + jy + zk}{r} = \frac{r}{r} = \hat{r}$

Question 2: If $r = xi + yj + zk$ then show that

$\text{grad } \log |r| = \frac{r}{r^2}$ $\left(\text{or } \nabla \log |r| = \frac{r}{r^2}\right)$

Solution: Since $r = xi + yj + zk$ we have

$$|r| = \sqrt{x^2 + y^2 + z^2}$$

$\therefore \log |r| = \log \sqrt{x^2 + y^2 + z^2}$

$= \frac{1}{2} \log (x^2 + y^2 + z^2)$

$\therefore \text{grad } \log |r| = \nabla \log |r| = \nabla \frac{1}{2} \log (x^2 + y^2 + z^2)$

$= \frac{1}{2} \nabla \log (x^2+y^2+z^2)$

$= \frac{1}{2} \sum i \frac{\partial}{\partial x} \log (x^2+y^2+z^2)$

$= \frac{1}{2} \sum i \frac{2x}{x^2+y^2+z^2}$

$= \frac{xi+yj+zk}{x^2+y^2+z^2}$

$= \frac{\vec{r}}{r^2} = \frac{\vec{r}}{r} \cdot \frac{1}{r} = \frac{\hat{r}}{r}$

Quest 3: If $\bar{r} = x\hat{i} + yj + zk$ then show that

$\text{grad} \left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2}$ $\left\{ \text{or } \nabla \left(\frac{1}{r} = -\frac{\hat{r}}{r^2}\right) \right\}$

Solution: $\text{grad} \left(\frac{1}{r}\right) = \nabla \left(\frac{1}{r}\right)$

$= i \frac{\partial}{\partial x}\left(\frac{1}{r}\right) + j \frac{\partial}{\partial y}\left(\frac{1}{r}\right) + k \frac{\partial}{\partial z}\left(\frac{1}{r}\right)$

$= -i \frac{1}{r^2} \frac{\partial r}{\partial x} - j \frac{1}{r^2} \frac{\partial r}{\partial y} - k \frac{1}{r^2} \frac{\partial r}{\partial z}$

$= -\frac{1}{r^2} \left\{ i \frac{x}{r} + j \frac{y}{r} + k \frac{z}{r} \right\}$ by Q① and

$= -\frac{xi+yj+zk}{r^3}$

$= \boxed{-\frac{\bar{r}}{r^3} = -\frac{\hat{r}}{r^2}}$

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4. If $r = xi + yj + zk$ Then show that

$$\text{grad } r^n = n r^{n-2} \bar{r}$$

or $\nabla r^n = n r^{n-2} \bar{r}$ where $|\bar{r}| = r$

Solution: Here $\text{grad } r^n = \nabla r^n = i \frac{\partial r^n}{\partial x} + j \frac{\partial r^n}{\partial y} + k \frac{\partial r^n}{\partial z}$

$$= i n r^{n-1} \frac{\partial r}{\partial x} + j n r^{n-1} \frac{\partial r}{\partial y} + k n r^{n-1} \frac{\partial r}{\partial z}$$

$$= n r^{n-1} \left\{ i \frac{x}{r} + j \frac{y}{r} + k \frac{z}{r} \right\} \text{ [by Q.1]}$$

$$= n r^{n-2} (xi + yj + zk)$$

$$= n r^{n-2} \bar{r}$$

Question 5: show that $\text{grad } f(r) \times \bar{r} = \bar{0}$

$$\text{grad } f(r) = \nabla f(r) = i \frac{\partial}{\partial x} f(r) + j \frac{\partial}{\partial y} f(r) + k \frac{\partial}{\partial z} f(r)$$

$$= i \frac{\partial f}{\partial r} \frac{\partial r}{\partial x} + j \frac{\partial f}{\partial r} \frac{\partial r}{\partial y} + k \frac{\partial f}{\partial r} \frac{\partial r}{\partial z}$$

$$= 2r e^{r^2} \left\{ i \frac{x}{r} + j \frac{y}{r} + k \frac{z}{r} \right\} \left[\because \frac{\partial r}{\partial x} = \frac{x}{r} \right]$$

$$= \frac{\partial f}{\partial r} \frac{xi + yj + zk}{r}$$

$$= \frac{1}{r} \frac{\partial f}{\partial r} \bar{r}$$

Hence $\text{grad } f(r) \times \bar{r} = \frac{1}{r} \frac{\partial f}{\partial r} \bar{r} \times \bar{r}$

$$= \bar{0} \qquad \because \bar{r} \times \bar{r} = 0$$

Q.6:

Evaluate grad $e^{(x^2+y^2+z^2)}$ or grad e^{r^2} or ∇e^{r^2}

Solution: since $x^2+y^2+z^2 = r^2$ we have

$$\text{grad}\, e^{(x^2+y^2+z^2)} = \nabla e^{r^2} = \left(i\frac{\partial}{\partial x} + J\frac{\partial}{\partial y} + K\frac{\partial}{\partial z}\right) e^{r^2}$$

$$= i e^{r^2} 2r \frac{\partial r}{\partial x} + J e^{r^2} 2r \frac{\partial r}{\partial y} + K e^{r^2} 2r \frac{\partial r}{\partial z}$$

$$= 2r e^{r^2}\left\{ i\frac{x}{r} + J\frac{y}{r} + K\frac{z}{r}\right\} \left[\because \frac{\partial r}{\partial x} = \frac{x}{r}\right]$$

$$= 2e^{r^2}(xi + yJ + zK)$$

$$= 2e^{r^2}\bar{r}$$

Q.7: If a and b are constant vectors then show that

$\nabla[r\, a\, b] = a\times b$ or grad $[r\, a\, b] = a\times b$

Solution: Let $\bar{a} = a_1 i + a_2 J + a_3 K$ and $\bar{b} = b_1 i + b_2 J + b_3 K$. Then

$$[r\, a\, b] = r\cdot(a\times b) = \begin{vmatrix} x & y & z \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)x + (a_3b_1 - a_1b_3)y + (a_1b_2 - a_2b_1)z$$

$$\therefore \nabla[r\, a\, b] = \text{grad}[r\, a\, b] = \left(i\frac{\partial}{\partial x} + J\frac{\partial}{\partial y} + K\frac{\partial}{\partial z}\right)[(a_2b_3 - a_3b_2)x + (a_3b_1 - a_1b_3)y + (a_1b_2 - a_2b_1)z]$$

$$= i(a_2b_3 - a_3b_2) + J(a_3b_1 - a_1b_3) + K(a_1b_2 - a_2b_1)$$

$$= \begin{vmatrix} i & J & K \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= a\times b$$

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Q. 8. Find the directional derivative of $\phi = xy + yz + zx$ in the direction of the vector $i + 2j + 2k$ at the point $(1, 2, 0)$

solution: $$\hat{a} = \frac{i+2j+2k}{\sqrt{1^2+2^2+2^2}} = \frac{i+2j+2k}{3}$$

Also $$\text{grad } \phi = i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}$$

$$= (y+z)i + (z+x)j + (x+y)k$$

$$\therefore \frac{d\phi}{ds} = \bar{a} \cdot \text{grad } \phi$$

$$= \frac{1}{3}(i+2j+2k)\cdot[(y+z)i + (z+x)j + (x+y)k]$$

$$= \frac{1}{3}[(y+z) + 2(z+x) + 2(x+y)] = \frac{1}{3}(4x+3y+3z)$$

$$= \frac{10}{3} \text{ at the point } (1, 2, 0)$$

Q. 9. Find the directional derivative of $\phi = x^2yz + 4xz^2$ in the direction of the vector $2i - j - 2k$ at the point $(1, -2, -1)$

Solution: Let $\hat{a}$ be the unit vector along the direction of $2i - j - 2k$. Then

$$\hat{a} = \frac{2i-j-2k}{\sqrt{2^2+(-1)^2+(-2)^2}} = \frac{2i-j-2k}{3}$$

Also $$\text{grad } \phi = \nabla\phi = i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}$$

$$= (2xyz + 4z^2)i + (x^2z)j + (x^2y + 8xz)k$$

Hence the directional derivative along the given direction

$$\frac{d\phi}{ds} = \hat{a} \text{ grad } \phi$$

$$= \frac{1}{3}(2i-j-2k)\cdot[(2xyz+4z^2)i + (x^2z)j + (x^2y+8xz)k]$$

$$= \frac{2}{3}(2xyz+4z^2) - \frac{1}{3}x^2z - \frac{2}{3}(x^2y+8xz)$$

$$= \frac{16}{3} + \frac{1}{3} + \frac{20}{2} = \frac{37}{3} \text{ at the point } (1,-2,-1)$$

Ans