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S. N.	Semester Registration No.	Name of the Student		Assignment
1	Y18272101	Ankit Kumar Thakur		properties of LPP
2	Y18272102	Anushri Jadiya		
3	Y18272104	Chandni Thakur		
4	Y18272105	Geeta Bai		
5	Y18272106	Hemlata Dangi		Game theory
6	Y18272107	Hitesh Mahore		
7	Y18272108	Kajal Sharma		
8	Y18272109	Kirti Gupta		
9	Y18272110	Lalit Kumar		Inventory modelling
10	Y18272112	Naresh Kumar Jhariya		
11	Y18272113	Nidhi Yadav		
12	Y18272114	Pupespandra Choudhar		
13	Y18272115	Ram Sagar		dynamic programming
14	Y18272116	Richa Jain		
15	Y18272118	Shivangi Sipolya		
16	Y18272119	Shivani Singh Raghuvanshi		
17	Y18272120	Taiyaba		Kuhn Tucker theorems
18	Y18272121	Vaishnavi Sharma		
19	Y18272124	Shalini Jain		
20	Y18272126	Nidhi Sharma		
21	Y18272127	Deepali Rajoriya		quardric programming
22	Y18272128	Chanrika Vaidhya		
23	Y18272130	Harshita Jain		

Inventory

An inventory can be defined as a stock of goods which must be carried out in order to ensure smooth and effecient running of business affairs.

Notations and definition

Holding cost:— The cost associated with carrying or holding the goods in stock is called holding cost denoted by c_1.

unit of Holding cost is c_1 unit per unit

Shortage Cost:— When the stock of items is embly and ordered quantity arrives, then the cost associated is penalty or shortage cost, denoted by c_2.

Setup Cost The costs associated with obtaining goods through purchasing or manufacturing or ordering are known as set up cost, denoted by c_3.

Q. Why inventory is maintained?

Model I(a) The Economic lot size system

To establish this model we state the following.

1) Demand is uniform at a rate R per time
2) lead time is zero
3) Production rate is infinite
4) shortage are not allowed.
5) holding cost is c_1 unit per unit time
6) Set up cost is c_3 per setup

Suppose quantity q is required ie quantity = Demand × time

$q = Rt$

So the total quantity $= \int_0^t Rt\,dt = \frac{1}{2}Rt^2 = \frac{1}{2}qt$

Holding cost $= c_1$ (quantity) $= c_1 \frac{1}{2} Rt^2$ —— ①

Shortage cost $= 0$ ($\because$ shortage are not required)

Set up cost $= C_3$

total cost $= \frac{1}{2} C_1 R t^2 + 0 + C_3$ (sum of holding cost + shortage cost + setup cost)

Average cost $= \frac{\text{total cost}}{t}$

$$C(t) = \frac{1}{2} C_1 R t + \frac{C_3}{t}$$

Now we have to find optimum cost (minimum cost)

$$\frac{d}{dt} C(t) = \frac{1}{2} C_1 R - \frac{C_3}{t^2}$$

$$\Rightarrow t = \sqrt{\frac{2C_3}{C_1 R}}$$

$$\frac{d^2}{dt^2} C(t) = \frac{2C_3}{t^3} > 0$$

i.e. $C(t)$ is minimum at $t^* = \sqrt{\frac{2C_3}{C_1 R}}$

$$C(t)_{\text{minimum}} = \frac{1}{2} C_1 R \left(\sqrt{\frac{2C_3}{C_1 R}}\right) + \frac{C_3}{\sqrt{\frac{2C_3}{C_1 R}}}$$

$$C_{\text{minimum}} = \sqrt{2 C_1 C_3 R}$$

$$q^* = R t^*$$

$$= \sqrt{\frac{2 C_3 R}{C_1}}$$

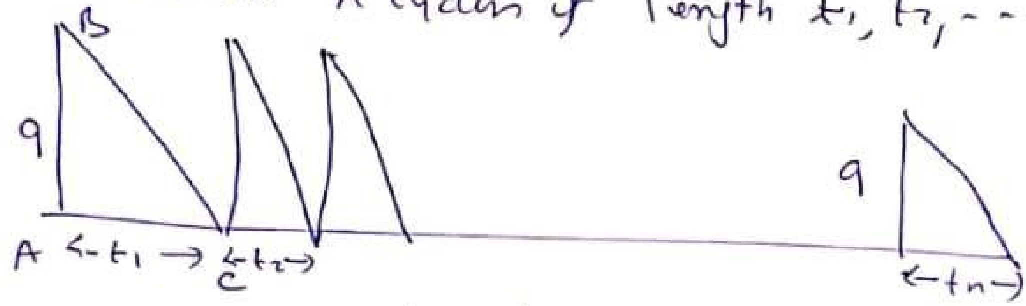
$$t^* = \sqrt{\frac{2C_3}{C_1 R}}$$

Model I(b) Economic lot size with different rates of demand in different cycle.

To establish this model we state

1) Total Demand D is constant
2) shortage not occured
3) Holding cost is C_1 unit per unit time
4) Setup cost C_3 per set up.
5) ~~Total~~ required quantity is q per cycle.
6) lead time in zero

Suppose there are n cycles of length $t_1, t_2, \ldots$ and t_n.



Holding cost $= C_1$ (Sum of areas of all traingles)

$$= C_1\left(\frac{1}{2}qt_1 + \frac{1}{2}qt_2 + \cdots + \frac{1}{2}qt_n\right)$$

$$= \frac{C_1}{2}q(t_1+t_2 \cdots + t_n)$$

$$= \frac{C_1}{2}qT$$

Setup cost $= C_3 \times$ number of cycles

$$= C_3 \times \frac{\text{Total Demand}}{\text{quantity required per cycles}}$$

$$= C_3 \times \frac{D}{q}$$

Total cost $C(q) = \frac{1}{2}C_1 q T + \frac{D}{q}C_3.$

This is your home work to find optimum $C(q)$ w.r.t q

one can easily get $q^* = \sqrt{\frac{2C_3(D/T)}{C_1}}$

$$C_{mini}(q^*) = \sqrt{2C_1C_2(D/T)}$$

Model I (c) Economic lot size with finite rate of replenishment

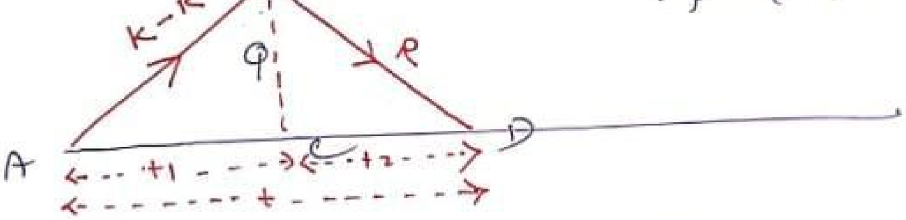
To establish this model we state-

1) Holding cost is C_1 unit per unit time (C_1)
2) Shortages are not allowed
3) Demand rate R ~~per~~ unit per time
4) production rate is infinite $(K>R)$;

પ્રોડક્શન રેટ K છે. [illegible] હોવાથી K>R હોય છે K ઉત્પાદન ઝડપી કરતે છે.

As per this model we start the production at rate K from time 0 to time t_1, since there will be demand rate is R unit per time, so inventory increases at the rate $K-R$ till time t_1.
After time t_1 to t_2 only demand occures (See fig 1)

$\because \varphi = (K-R)t_1$



Holding cost $= C_1$ (Area of triangle ABC & BCD)

$$= C_1\left(\tfrac{1}{2}\varphi t\right)$$

Setup cost $= C_3$.

Average cost $\quad C(q) = \dfrac{\frac{1}{2}\varphi t C_1}{t} + \dfrac{C_3}{t}$

Since $\varphi = (K-R)t_1$, $\varphi = Rt_2$, $q = Rt$

The total quality required is q and consumption consumed in time t_2 is Rt_2. ie $q = \varphi + Rt_2$

ie $\varphi = q - Rt_2$ —— (11)

① & (11) $\Rightarrow \quad \varphi = \dfrac{K-R}{K}\cdot q$

Now $\quad C(q) = \frac{1}{2}\varphi C_1 + \dfrac{C_3}{t}$

$$C(q) = \frac{1}{2}\frac{K-R}{K}\cdot q\, C_1 + \frac{C_3 R}{q}.$$

This is how we to find optimum $C(q)$ w.r.t. q.

$$q^* = \sqrt{\frac{2C_3}{C_1}\cdot\frac{RK}{K-R}}$$

$$t^* = \frac{q^*}{R}$$

$$C^*_{minim}(q) = \sqrt{2C_1\left(1-\frac{R}{K}\right)C_3 R}.$$

Unit 5.

Dynamic Programming: Dynamic programming is a mathematical technique dealing with the optimization of multistage decision processes. In dynamic programming a large problem is splitted into sub-problem, each of which involves only a few variables.

Bellman's principle of optimality   Model-I

= An optimal policy has the property that whatever the initial state and decision are the remaining decisions must constitute an optimal policy with regards to the state resulting from the first decision.

Consider the implication of this principle as a multi stage decision problem. The problem which does not satisfy the principle of optimality. Aim is to find optimal return path-

$f_N(x)$ = the optimal return due to decision d_n

$r(d_n)$ = immediate function due to decision d_n

$T(x, d_n)$ = the transfer function which gives the resulting state

$\{x\}$ = set of admissible decisions.

Then the optimal return will be

$$f_N(x) = \max_{d_n \in \{x\}} \left[r(d_n) + f_{N-1}\{T(x, d_n)\} \right].$$

Model-II   Single additive constraint, multiplicatively seperable return.

Consider the problem: To maximize $Z = \prod_{j=1}^{n} f_j(y_j)$

subjected to $\sum_{j=1}^{n} a_j y_j = b$, $y_j \geq 0$, $a_j \geq 0$.

Solution   First to introduce the variable

$a_1 y_1 = s_1$

$a_1 y_1 + a_2 y_2 = s_2$

$\vdots$

$a_1 y_1 + \cdots + a_n y_n = s_n$

and $F_j(s_j) = \max_{y_1, y_2 \cdots y_j} \prod_{j=1}^{j} f_j(y_j)$, $2 \le j \le n$.

Case I

$n=1$ $F_1(s_1) = f_1(y_1)$ when $a_1 y_1 = s_1$

$= f_1\left(\frac{s_1}{a_1}\right)$

$n=2$ $F_2(s_2) = \max f_1(y_1) \cdot f_2(y_2)$, $a_1y_1 + a_2y_2 = s_2$

$= \max f_2(y_2) \cdot F_1(s_1)$

$= \max f_2(y_2)\, f_1\left(\frac{s_1}{a_1}\right)$

for j $F_j(s_j) = \max_{y_1 \cdots y_j} f_1(y_1) \cdots f_j(s_j)$

when $a_1y_1 + \cdots + a_jy_j = s_j$

for n $F_n(s_n) = \max_{y_1 \cdots y_n} f_1(y_1) \cdots f_n(y_n)$

when $a_1y_1 + \cdots + a_ny_n = s_n$

Example 3 find the value of maxi $(y_1 \cdot y_2 \cdot y_3)$

subjected to $y_1 + y_2 + y_3 = 5$, $y_1, y_2, y_3 \geq 0$

Solution (we have to solve above problem by inductive method step by step)

For $n=1$, $F_1(s_1) = y_1$. where $y_1 = s_1 = 5$.

$= s_1$

For $n=2$, $F_2(s_2) = \max y_1 \cdot y_2$ when $y_1 + y_2 = s_2$

$= \max y_2 \cdot \{y_1\}$

$= \max y_2 \{F_1(s_1)\}$ $\because F_1(s_1) = s_1 = y_1$

$= \max y_2 \{y_1\}$

$= \max (s_2 - y_2) \cdot y_2$

Now to find maximum of $f = (s_2 - y_2) \cdot y_2 = s_2y_2 - y_2^2$

$f' = s_2 - 2y_2 = 0$

i.e $y_2 = \frac{s_2}{2}$

$f'' = -ve$ at point $y_2 = \frac{s_2}{2}$

hence f is maximum at point $y_2 = s_2$

i.e. $f_{maxi} = \left(s_2 - \frac{s_2}{2}\right)\frac{s_2}{2}$

$$= \left(\frac{s_2}{2}\right)^2$$

or $f_2(s_2) = \max\ (s_2 - y_2)\, y_2$

$$= \left(\frac{s_2}{2}\right)^2 \quad \text{when } y_1 + y_2 = s_2$$

<u>for $n = 3$.</u>

$$f_3(s_3) = \max\ y_1 \cdot y_2 \cdot y_3$$

$$= \max\ f_2(s_2) \cdot y_3$$

$$= \max \left(\frac{s_2}{2}\right)^2 \cdot y_3 \qquad \because y_1 + y_2 + y_3 = s_3 = 5$$

$$\text{or } s_2 + y_3 = s_3 = 5$$

$$= \max\ \frac{(s_3 - y_3)^2}{4}\, y_3$$

this is a known work to find maxi of $g = \frac{(s_3 - y_3)^2}{4} \cdot y_3$ using differentially $g' = 0$ & $g'' =$ -ve.

one can easily find above is maximum at $y_3 = \frac{s_3}{3}$.

then $f_3(s_3) = \dfrac{\left(s_3 - \frac{s_3}{3}\right)^2}{4} \cdot \dfrac{s_3}{3}$

$$= \left(\frac{s_3}{3}\right)^3$$

$$= \left(\frac{5}{3}\right)^3$$

$$= \frac{125}{27} \quad \#$$

Gamble 4 : optimal sub division problem

Devide a quantity b into n parts so as to maximize their product.

$$f_1(b) = b$$

$$f_n(b) = \max_{0 \le z \le b} z\, f_{n-1}(b-z)$$

hence find $f_n(b)$ and the division that maximizes it.

Solution for $n=1$ $\quad f_1(S_1) = y_1$ when $y_1 = S_1 = b$

for $n=2$ $\quad f_2(S_2) = \underset{\text{maximize}}{y_1 \cdot y_2}$ when $y_1 + y_2 = S_2$

$F_3(S_3) =$ maximum $y_1 \cdot y_2 \cdot y_3$

when $y_1 + y_2 + y_3 = S_3$

Ans: $f_n(s_n) = \left(\frac{b}{n}\right)^{n+1}$

procedure as per example 3. (Home work)

serial No. of example are as per S.D. Sharma

Model III: Single additive constraint, additive seperable return (Home work)

Example 8 minimize $z = y_1^2 + y_2^2 + y_3^2$.

subjected to $y_1 + y_2 + y_3 \ge 15$, $y_1, y_2, y_3 \ge 0$.

(Home work)

Example 9. Use dynamic programing to show that

$\sum_{i=1}^{n} p_i \log p_i$ subjected to condition $\sum_{i=1}^{n} p_i = 1$

is minimum when $p_1 = p_2 \cdots = p_n = \frac{1}{n}$

Solution. For $n=1$, $\quad F_1(S_1) = p_1 \log p_1$ when $p_1 = S_1 = 1$

$= 1 \log 1$

For $n=2$, $\quad F_2(S_2) = \underset{\text{mini}}{} p_1 \log p_1 + p_2 \log p_2$

when $p_1 + p_2 = S_2 = 1$

⑤

$$F_2(s_2) = \underset{mini}{} \; p_1 \log p_1 + p_2 \log p_2 \qquad \because p_1 + p_2 = s_2 = 1$$

$$= \underset{mini}{} \; (1-p_2)\log(1-p_2) + p_2 \log p_2$$

Here $f = (1-p_2)\log(1-p_2) + p_2 \log p_2$ (say).

$f' = -1 - \log(1-p_2) + 1 + \log p_2 = 0$

$\Rightarrow p_2 = \frac{1}{2}$

$f'' = \oplus$ve at point $p_2 = \frac{1}{2}$

ie $p_1 = p_2 = 1$

$$F_2(s_2) = 2\left(\frac{1}{2} \log \frac{1}{2}\right) \; ; \quad p_1 = p_2 = \frac{1}{2}$$

for $n=3$ $\quad F_3(s_3) = 3\left(\frac{1}{3} \cdot \log \frac{1}{3}\right)$, $\; p_1 = p_2 = p_3 = \frac{1}{3}$

$\vdots$

$$F_j(s_j) = j\left(\frac{1}{j} \log \frac{1}{j}\right); \; p_1 = p_2 \cdots = p_j = \frac{1}{j}$$

$\vdots$

$$F_n(s_n) = n\left(\frac{1}{n} \log \frac{1}{n}\right) \text{ on } p_1 = p_2 = \cdots = p_n = \frac{1}{n}$$

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one can extend for $n = m+1$

ie $\quad F_{m+1}(s_{m+1}) = (m+1)\frac{1}{m+1} \log \frac{1}{m+1}$; $p_1 = p_2 \cdots = p_m = p_{m+1} = \frac{1}{m+1}$

Hence $\quad F_n(s_n) = n\left(\frac{1}{n} \log \frac{1}{n}\right)$; $p_1 = p_2 \cdots = p_n = \frac{1}{n}$

Ex 15 $\quad$ minimize $Z = y_1 + y_2 + \cdots + y_n$

such $\quad y_1 \cdot y_2 \cdot y_3 \cdots y_n = d \;$; $y_j \geq 0$
